Generalized Nonlinear Complementary Attitude Filter

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I. Introduction

The multiplicative extended Kalman filter (MEKF) is a proven attitude estimation technique[1, 2], that is the *de facto* standard for low-cost inertial measurement units (IMUs). However, the MEKF does have its drawbacks. It is computationally expensive, difficult to tune, and potentially subject to divergence due to numerical errors. This has lead to the development of alternate attitude estimation algorithms such as the unscented Kalman filter [3, 4], generalized MEKF [5], invariant extended Kalman filter [6], particle filters [7, 8], and a host of nonlinear observers [9–16]. These techniques and others are summarized in a review article [17] by Crassidis *et al.*.

A relatively new and popular estimation technique is Mahony's nonlinear complementary filter [13, 18]. This filter boasts computational efficiency, a small number of easily tuned, intuitive parameters, and a proof of almost global asymptotic stability. Moreover, this filter has been shown to perform similarly to the traditional MEKF [19].

This work describes a new attitude estimation technique that is a generalization of Mahony's nonlinear complementary filter. Interestingly, this generalized attitude filter shares a close mathematical relationship with the MEKF, and indeed both the MEKF without gyro bias correction and the constant gain MEKF with gyro bias correction may be viewed as special cases of the generalized attitude filter.

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II. Attitude Estimation

The attitude estimation problem consists of combining measurements from various potentially imperfect sensors located onboard an object (*e.g.* vehicle, aircraft, *etc*...) into an accurate estimate of the attitude of the object. There are many versions of this problem, which vary in parameters such as the attitude representation, object dynamics, and measurement models.

The two primary attitude representations used in this work are the direction cosine matrices for a global representation and the Euler vector for small changes in attitude near the identity. The global attitude, meaning the rotation from the inertial frame to the frame that rotates with the object (*i.e.* body frame), is denoted by C. From another viewpoint, C represents the transformation from body to inertial coordinates: $C\boldsymbol{v} = \boldsymbol{v}^{\mathcal{I}}$. (As described in the Appendix, the notation $^{\mathcal{I}}$ denotes vectors or matrices in inertial coordinates; otherwise, vectors or matrices should be assumed to be in body coordinates.) Small changes in attitude are denoted by $\tilde{C}$ or by the Euler vector $\boldsymbol{x}$ where $\tilde{C} \approx I + \boldsymbol{x}_\times$.

No assumptions are made as to the dynamics of the rotating object. Rather, the process equations are based on the kinematics of the rotation group itself. Specifically, the attitude C of an object with angular velocity $\boldsymbol{\omega}$, measured in the body frame, evolves according to

$$\dot{C} = C\boldsymbol{\omega}_\times \tag{1}$$

The generality of the kinematics allows these results to apply to a wide range of attitude estimation problems.

Finally, this work assumes a typical set of measurements, including an angular rate measurement from a 3-axis rate gyro and a set of vector measurements from, for example, a 3-axis magnetometer measuring Earth's magnetic field and a 3-axis accelerometer measuring, with appropriate filtering, Earth's gravity vector. The angular rate measurement, $\check{\boldsymbol{\omega}}$, is corrupted from the true angular rate, $\boldsymbol{\omega}$, by additive zero-mean white noise, $\boldsymbol{\eta}_\omega$, and a slowly varying bias error, $\boldsymbol{b}$, which is driven by a white noise process, $\boldsymbol{\eta}_b$. Similarly, the vector measurements, $\check{\boldsymbol{v}}_n$, of inertial vectors in the body frame are corrupted from the true vectors, $C^T\boldsymbol{v}_n^{\mathcal{I}}$ by additive white noise, $\boldsymbol{\eta}_{v_n}$. Finally, it is assumed that the vector measurements have all been normalized to be unit vectors. To summarize, the angular

rate and vector measurement models used in this work are

$$\breve{\boldsymbol{\omega}} = \boldsymbol{\omega} + \boldsymbol{b} + \boldsymbol{\eta}_\omega, \quad \dot{\boldsymbol{b}} = \boldsymbol{\eta}_b \tag{2a}$$

$$\breve{\boldsymbol{v}}_n = C^T \boldsymbol{v}_n^{\mathcal{I}} + \boldsymbol{\eta}_{v_n}, \quad |\boldsymbol{v}_n^{\mathcal{I}}| = 1 \tag{2b}$$

The kinematics (1) and measurement model (2) completely specify the attitude estimation problem.

III. Multiplicative Extended Kalman Filter

The MEKF applies the extended Kalman filter (EKF) formalism [20] to the attitude estimation problem. Although the MEKF typically uses quaternions as its attitude representation, it is reformulated here in terms of direction cosine matrices and Euler vectors to facilitate comparison with the nonlinear complementary filter. After the derivation of the process equations for the Euler vector, this description of the MEKF closely follows the form given in [2].

The true attitude C, estimated attitude C_{ref}, and error in attitude $\tilde{C}$ are related according to

$$\tilde{C} = C_{\text{ref}}^T C = I + \boldsymbol{x}_\times + \dots \tag{3}$$

where $\tilde{C}$ is linearized about the identity with the three-component Euler vector, $\boldsymbol{x}$, as discussed earlier. The process equations for the Euler vector are determined by the kinematics of the rotation group (1). These kinematics apply equally well to the reference attitude: $\dot{C}_{\text{ref}} = C_{\text{ref}}(\boldsymbol{\omega}_{\text{ref}})_\times$, where $\boldsymbol{\omega}_{\text{ref}}$ is the angular velocity of the reference attitude. Thus,

$$\begin{aligned}
\dot{\tilde{C}} &= \dot{C}_{\text{ref}}^T C + C_{\text{ref}}^T \dot{C} \\
&= -(\boldsymbol{\omega}_{\text{ref}})_\times \tilde{C} + \tilde{C}\boldsymbol{\omega}_\times \\
&= (\boldsymbol{\omega} - \boldsymbol{\omega}_{\text{ref}})_\times + \left[\boldsymbol{x}_\times, \frac{1}{2}(\boldsymbol{\omega} + \boldsymbol{\omega}_{\text{ref}})_\times\right] \\
&\quad + \left\{\boldsymbol{x}_\times, \frac{1}{2}(\boldsymbol{\omega} - \boldsymbol{\omega}_{\text{ref}})_\times\right\} + \dots
\end{aligned}$$

The symmetric terms are second or higher order in $\boldsymbol{x}_\times$. Retaining only the first-order anti-symmetric terms of (3) yields the equation for the evolution of the Euler vector:

$$\dot{\boldsymbol{x}} \approx \boldsymbol{\omega} - \boldsymbol{\omega}_{\text{ref}} - \frac{1}{2}(\boldsymbol{\omega} + \boldsymbol{\omega}_{\text{ref}}) \times \boldsymbol{x}. \tag{4}$$

It is now possible to apply the EKF formalism to form an estimate, $\hat{\boldsymbol{x}}$, of this attitude error vector. Following the typical procedure for the MEKF, $\boldsymbol{\omega}_{\text{ref}}$ is set such that $\langle\hat{\boldsymbol{x}}\rangle = \boldsymbol{0}$. Thus, $\boldsymbol{\omega}_{\text{ref}}$

represents an estimate of the error between the current reference attitude C_{ref} and the true attitude, C, which may now be estimated simply by integrating (1) with $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathrm{ref}}$. Finally, similar to the results in [2], the defining equations for the continuous-time MEKF are:

$$\boldsymbol{\omega}_{\mathrm{ref}} = \check{\boldsymbol{\omega}} - \hat{\boldsymbol{b}} + P_a \sum_n \sigma_{v_n}^{-2} \left(\check{\boldsymbol{v}}_n \times \boldsymbol{v}_{\mathrm{ref}\,n} \right) \tag{5a}$$

$$\dot{\hat{\boldsymbol{b}}} = P_c^T \sum_n \sigma_{v_n}^{-2} \left(\check{\boldsymbol{v}}_n \times \boldsymbol{v}_{\mathrm{ref}\,n} \right) \tag{5b}$$

where $\boldsymbol{v}_{\mathrm{ref}\,n} = C_{\mathrm{ref}}^T \boldsymbol{v}_n^{\mathcal{I}}$ and σ_{v_n} is the measurement noise covariance based on $\boldsymbol{\eta}_{v_n}$. Also, the covariance matrix, P, has been partitioned into 3×3 blocks

$$P = \begin{bmatrix} P_a & P_c \\ P_c^T & P_b \end{bmatrix}$$

with P_a representing the attitude covariances and P_b representing the bias covariances. The covariance matrix evolves according to the standard Ricatti equation, which becomes

$$\begin{aligned} \dot{P} = & \begin{bmatrix} [P_a, \boldsymbol{\omega}'_\times] - 2\mathbb{P}_s(P_c) & -\boldsymbol{\omega}'_\times P_c - P_b \\ P_c^T \boldsymbol{\omega}'_\times - P_b & 0 \end{bmatrix} \\ & + \begin{bmatrix} \sigma_\omega^2 I & 0 \\ 0 & \sigma_b^2 I \end{bmatrix} \\ & + \sum_n \sigma_{v_n}^{-2} \begin{bmatrix} P_a (\boldsymbol{v}_{\mathrm{ref}\,n})_\times^2 P_a & P_a (\boldsymbol{v}_{\mathrm{ref}\,n})_\times^2 P_c \\ P_c^T (\boldsymbol{v}_{\mathrm{ref}\,n})_\times^2 P_a & P_c^T (\boldsymbol{v}_{\mathrm{ref}\,n})_\times^2 P_c \end{bmatrix} \end{aligned} \tag{6}$$

where $\boldsymbol{\omega}' = \frac{1}{2}(\check{\boldsymbol{\omega}} + \boldsymbol{\omega}_{\mathrm{ref}} - \hat{\boldsymbol{b}})$ and σ_ω^2 and σ_b^2 are process noise covariances based on $\boldsymbol{\eta}_\omega$ and $\boldsymbol{\eta}_b$ respectively.

For reasons discussed in Section IV below, the full continuous-time MEKF is not a special case of the generalized attitude filter. However, two important special cases of the MEKF are related to the generalized attitude filter. These are the bias-free MEKF and the constant gain MEKF. The bias-free MEKF simply assumes that the gyro measurement has no bias term, $\hat{\boldsymbol{b}} = \boldsymbol{0}$. The constant gain MEKF, which is often used to reduce computational load, retains the gyro bias term but assumes that the covariance matrix is constant at the value it would approach as $t \to \infty$ with $\boldsymbol{\omega} = \boldsymbol{0}$.

IV. Generalized Nonlinear Complementary Attitude Filter

The nonlinear complementary filters are a set of attitude estimators inspired by traditional linear complementary filters. Like a linear complementary filter, the nonlinear complementary filters combine low frequency attitude information from a set of vector measurements with high frequency attitude information from rate gyros. An interesting feature of this family of estimators is the ability to prove the almost global asymptotic stability of the estimator. This work demonstrates how the stability proof may be extended to a generalization of the nonlinear complementary filters. Moreover, it is shown that the deterministic [21] forms of the MEKF without gyro biases and the constant gain MEKF with gyro biases are special cases of this generalized attitude filter, and thus, it is proven that these forms of the MEKF are almost globally asymptotically stable.

The form for Mahony's explicit complementary filter [13] is

$$\dot{\hat{C}} = \hat{C}\left(\breve{\boldsymbol{\omega}} - \hat{\boldsymbol{b}} + k_P \boldsymbol{\omega}_{\text{err}}\right)_\times, \quad k_P > 0 \tag{7a}$$

$$\dot{\hat{\boldsymbol{b}}} = -k_I \boldsymbol{\omega}_{\text{err}}, \qquad k_I > 0 \tag{7b}$$

$$\boldsymbol{\omega}_{\text{err}} = \sum_n k_n \breve{\boldsymbol{v}}_n \times \hat{\boldsymbol{v}}_n, \qquad k_n > 0 \tag{7c}$$

where $\hat{\boldsymbol{v}}_n \equiv \hat{C}^T \boldsymbol{v}_n^{\mathcal{I}}$. This filter may be generalized simply by replacing the positive constant scalar gains, k_P and k_I, by potentially time-varying positive-definite matrix gains, K_P and K_I.

$$\dot{\hat{C}} = \hat{C}\left(\breve{\boldsymbol{\omega}} - \hat{\boldsymbol{b}} + K_P \boldsymbol{\omega}_{\text{err}}\right)_\times, \quad K_P > 0 \tag{8a}$$

$$\dot{\hat{\boldsymbol{b}}} = -K_I \boldsymbol{\omega}_{\text{err}}, \qquad K_I > 0 \tag{8b}$$

$$\boldsymbol{\omega}_{\text{err}} = \sum_n k_n \breve{\boldsymbol{v}}_n \times \hat{\boldsymbol{v}}_n, \qquad k_n > 0 \tag{8c}$$

At this point, it is interesting to note that the $(\breve{\boldsymbol{\omega}} - \hat{\boldsymbol{b}}) + K_P \boldsymbol{\omega}_{\text{err}}$ term from this filter is identical to the $\boldsymbol{\omega}_{\text{ref}}$ term (5a) from the MEKF, including the positive-definite nature of the covariance matrix. Moreover, the equation for $\dot{\hat{\boldsymbol{b}}}$ here is similar to (5b), the equation for the derivative of the bias estimate in the MEKF. However, an important distinction is that the integral matrix gain, $-P_c^T$, in the MEKF is not necessarily positive definite.

It is easier to analyze the stability properties of the error dynamics of the filter rather than the filter itself. Thus, the following definitions for the errors between the true and estimated attitudes

and biases are introduced.

$$\tilde{C} = \hat{C}^T C \tag{9a}$$

$$\tilde{\boldsymbol{b}} = \boldsymbol{b} - \hat{\boldsymbol{b}} \tag{9b}$$

The definition for the attitude error, $\tilde{C}$, here is the same as the definition used in the MEKF. By these definitions, the filter has converged to true attitude and bias when $(\tilde{C}, \tilde{\boldsymbol{b}}) = (I, \mathbf{0})$.

Combining the definitions for the error terms (9), the definition of the generalized attitude filter (8), and the measurement models (2) with no noise terms yields the equations for the error dynamics:

$$\dot{\tilde{C}} = [\tilde{C}, \boldsymbol{\omega}_\times] - (\tilde{\boldsymbol{b}} + K_P \boldsymbol{\omega}_{\text{err}})_\times \tilde{C} \tag{10a}$$

$$\dot{\tilde{\boldsymbol{b}}} = K_I \boldsymbol{\omega}_{\text{err}} \tag{10b}$$

$$\boldsymbol{\omega}_{\text{err}} = \sum_n k_n \boldsymbol{v}_n \times \tilde{C} \boldsymbol{v}_n \tag{10c}$$

As demonstrated in the proof below, the equilibria of the error dynamics, denoted $\tilde{C}_*$, are determined by the inertial vectors $\boldsymbol{v}_n^{\mathcal{I}}$ and the weights k_n, or more precisely by the matrix M:

$$M = C^T M^{\mathcal{I}} C \quad \text{with} \quad M^{\mathcal{I}} \equiv \sum_n k_n \boldsymbol{v}_n^{\mathcal{I}} (\boldsymbol{v}_n^{\mathcal{I}})^T. \tag{11}$$

The stability proof relies on M being positive semi-definite with distinct eigenvalues, which as shown in [13], is true if there are at least two non-parallel measurement vectors. Intuitively, the equilibria occur when the attitude error, $\tilde{C}$, is the identity rotation (*i.e.* the filter has converged) or a rotation of π rad about one of the principle axes of M. More specifically, the equilibria occur when $\mathbb{P}_a(\tilde{C}M) = 0$, which according to Lemma 1 below, implies that the equilibria are given by $\tilde{C}_{*i} \equiv U D_i U^T$ where $D_0 = I$, $D_1 = \text{diag}(1, -1, -1)$, $D_2 = \text{diag}(-1, 1, -1)$, and $D_3 = \text{diag}(-1, -1, 1)$ for $M = U \Lambda U^T$ with diagonal Λ and orthogonal U. The proof below first demonstrates that the equilibria indeed occur at $\tilde{C}_{*i}$, and then analyzes the stability properties of each equilibrium.

The following proof of the stability characteristics of the generalized attitude filter is substantially similar to the proofs given in [13]. The primary difference is the extension of the proof to handle potentially time-varying matrix gains rather than constant scalar gains. Some results from [13], such as the following lemma, transfer with no modification, and consequently are simply re-

stated here. The stability properties of the generalized attitude filter are now characterized with the following theorems.

Lemma 1. *Suppose $\tilde{C} \in SO(3)$ and M is positive semi-definite with distinct eigenvalues and decomposition $M = U\Lambda U^T$ for orthogonal U and diagonal Λ. Then, $\mathbb{P}_a(\tilde{C}M) = 0$ if and only if $\tilde{C} = \tilde{C}_{*i} \equiv UD_iU^T$ where $D_0 = I$, $D_1 = \mathrm{diag}(1,-1,-1)$, $D_2 = \mathrm{diag}(-1,1,-1)$, and $D_3 = \mathrm{diag}(-1,-1,1)$. [13]*

Theorem 1 (Stability of the generalized attitude filter)**.** *Consider the error dynamics described by (10). Suppose that K_P, K_I, and M are positive definite, $\dot{K}_I$ is positive semi-definite, and $k_n > 0$. Further, suppose that K_P and K_I are upper and lower bounded by positive constants; $\dot{K}_P$, $\dot{K}_I$, $\ddot{K}_I$, and $\boldsymbol{\omega}$ are bounded; and M has distinct eigenvalues. Then, the equilibrium point $(\tilde{C}, \tilde{\boldsymbol{b}}) = (I, \mathbf{0})$ of the error dynamics is asymptotically stable with a domain of attraction $\mathcal{D} = \left\{(\tilde{C}, \tilde{\boldsymbol{b}}) \in SO(3) \times \mathbb{R}^3 - \{(\tilde{C}_{*i}, \mathbf{0}) \,|\, i = 1,2,3\}\right\}$ and is locally exponentially stable.*

Proof. Consider the Lyapunov function candidate

$$v = \sum_n k_n - \mathrm{tr}(\tilde{C}M) + \frac{1}{2}\tilde{\boldsymbol{b}}^T K_I^{-1} \tilde{\boldsymbol{b}} \tag{12}$$

Following the standard procedure, the time derivative of v is calculated:

$$\begin{aligned}
\dot{v} &= -\mathrm{tr}(\dot{\tilde{C}}M + \tilde{C}\dot{M}) + \tilde{\boldsymbol{b}}^T K_I^{-1}\dot{\tilde{\boldsymbol{b}}} + \frac{1}{2}\tilde{\boldsymbol{b}}^T \frac{\mathrm{d}}{\mathrm{d}t} K_I^{-1}\tilde{\boldsymbol{b}} \\
&= -\mathrm{tr}\left([\tilde{C}M, \boldsymbol{\omega}_\times] - (\tilde{\boldsymbol{b}} + K_P\boldsymbol{\omega}_{\mathrm{err}})_\times \tilde{C}M\right) \\
&\quad + \tilde{\boldsymbol{b}}^T\boldsymbol{\omega}_{\mathrm{err}} - \frac{1}{2}\tilde{\boldsymbol{b}}^T K_I^{-1}\dot{K}_I K_I^{-1}\tilde{\boldsymbol{b}} \\
&= \mathrm{tr}\left((\tilde{\boldsymbol{b}} + K_P\boldsymbol{\omega}_{\mathrm{err}})_\times \mathbb{P}_a(\tilde{C}M)\right) - \frac{1}{2}\mathrm{tr}\left(\tilde{\boldsymbol{b}}_\times(\boldsymbol{\omega}_{\mathrm{err}})_\times\right) \\
&\quad - \frac{1}{2}\tilde{\boldsymbol{b}}^T K_I^{-1}\dot{K}_I K_I^{-1}\tilde{\boldsymbol{b}} \\
&= \mathrm{tr}\left((K_P\boldsymbol{\omega}_{\mathrm{err}})_\times \mathbb{P}_a(\tilde{C}M)\right) - \frac{1}{2}\tilde{\boldsymbol{b}}^T K_I^{-1}\dot{K}_I K_I^{-1}\tilde{\boldsymbol{b}}
\end{aligned} \tag{13}$$

where the identity (24c) from the Appendix and the result $(\boldsymbol{\omega}_{\mathrm{err}})_\times = 2\mathbb{P}_a(\tilde{C}M)$, also obtainable from the identities (24), were used in the simplification. As $\dot{K}_I \geq 0$, the second term in (13) is negative semi-definite. To show that the first term is also negative semi-definite, it is useful to

rewrite $K_P\boldsymbol{\omega}_{\text{err}}$ in terms of the measurement vectors

$$\begin{aligned}(K_P\boldsymbol{\omega}_{\text{err}})_\times &= \left(\sum_n k_n(Q\boldsymbol{v}_n)\times(Q\tilde{C}\boldsymbol{v}_n)\right)_\times \\ &= \sum_n k_n\left(Q\tilde{C}\boldsymbol{v}_n(\boldsymbol{v}_n)^TQ^T - Q\boldsymbol{v}_n(\tilde{C}\boldsymbol{v}_n)^TQ^T\right) \\ &= 2Q\mathbb{P}_a(\tilde{C}M)Q^T\end{aligned}$$

where $Q = \pm\sqrt{\det K_P}K_P^{-1}$ is a positive or negative definite matrix. Then, using the fact that Q has a decomposition $Q = S^T\Lambda S$ for orthogonal S and diagonal Λ:

$$\begin{aligned}&\operatorname{tr}\left((K_P\boldsymbol{\omega}_{\text{err}})_\times\mathbb{P}_a(\tilde{C}M)\right) \\ &\quad= 2\operatorname{tr}\left(Q\mathbb{P}_a(\tilde{C}M)Q^T\mathbb{P}_a(\tilde{C}M)\right) \\ &\quad= 2\operatorname{tr}\left((S^T\Lambda S)\mathbb{P}_a(\tilde{C}M)(S^T\Lambda S)\mathbb{P}_a(\tilde{C}M)\right) \\ &\quad= 2\operatorname{tr}\left(\Lambda(S\mathbb{P}_a(\tilde{C}M)S^T)\Lambda(S\mathbb{P}_a(\tilde{C}M)S^T)\right) \\ &\quad= 2\operatorname{tr}\left((\Lambda\mathbb{P}_a(S\tilde{C}MS^T))^2\right) \leq 0\end{aligned} \tag{14}$$

This results in the trace of the square of the product of a diagonal matrix and an antisymmetric matrix, which is easily shown to be negative semi-definite. Thus, combining (14) and (13), it is shown that $\dot{v} \leq 0$. Finally, with the assumptions on the boundedness of K_P, K_I, $\dot{K}_P$, $\dot{K}_I$, $\ddot{K}_I$, and $\boldsymbol{\omega}$, it is straightforward to show that $\ddot{v}$, given by

$$\begin{aligned}\ddot{v} =& 4\operatorname{tr}\left(Q\mathbb{P}_a(\tilde{C}M)\left(\dot{Q}\mathbb{P}_a(\tilde{C}M)\right.\right. \\ &\left.\left.+Q\mathbb{P}_a([\tilde{C}M,\boldsymbol{\omega}_\times] - 2Q\mathbb{P}_a(\tilde{C}M)Q^T\tilde{C}M)\right)\right) \\ &+\tilde{\boldsymbol{b}}^T\frac{\mathrm{d}}{\mathrm{d}t}K_I^{-1}\dot{\tilde{\boldsymbol{b}}} + \frac{1}{2}\tilde{\boldsymbol{b}}^T\frac{\mathrm{d}^2}{\mathrm{d}t^2}K_I^{-1}\tilde{\boldsymbol{b}}\end{aligned}$$

is bounded. Therefore, Barbalat's lemma [22] implies that $\dot{v}$ and thus $\mathbb{P}_a(\tilde{C}M)$ tend asymptotically to zero. According to Lemma 1, the attitude error, $\tilde{C}$ must approach one of the equilibria $\tilde{C}_{*i}$. Moreover, $\mathbb{P}_a(\tilde{C}M) \to 0$ implies that $\boldsymbol{\omega}_{\text{err}} \to 0$, and thus, after substituting the relation $\dot{\tilde{C}}_{*i} = [\tilde{C}_{*i},\boldsymbol{\omega}_\times]$ into (10), it is shown that $\tilde{\boldsymbol{b}}$ must also tend asymptotically to zero.

To demonstrate the stability characteristics of the various equilibria, the system is linearized about each $\tilde{C}_{*i}$. Because the equilibria attitude errors are constant in the inertial frame (*i.e.* $\dot{\tilde{C}}^{\mathcal{I}}_{*i} = 0$), it is easiest to conduct the linearization in the inertial frame. The error dynamics (10)

expressed in the inertial frame are

$$\dot{\tilde{C}}^{\mathcal{I}} = -(\tilde{\boldsymbol{b}}^{\mathcal{I}} + K_P^{\mathcal{I}}\boldsymbol{\omega}_{\text{err}}^{\mathcal{I}})_\times \tilde{C}^{\mathcal{I}} \tag{15a}$$

$$\dot{\tilde{\boldsymbol{b}}}^{\mathcal{I}} = \boldsymbol{\omega}_\times^{\mathcal{I}}\tilde{\boldsymbol{b}}^{\mathcal{I}} + K_I^{\mathcal{I}}\boldsymbol{\omega}_{\text{err}}^{\mathcal{I}} \tag{15b}$$

where $\tilde{C}^{\mathcal{I}} = C\tilde{C}C^T$ as described in the Appendix. Let $\tilde{C}^{\mathcal{I}} \approx \tilde{C}_{*i}^{\mathcal{I}}(1 + \boldsymbol{x}_\times^{\mathcal{I}})$ and $\tilde{\boldsymbol{b}}^{\mathcal{I}} \approx -\boldsymbol{y}^{\mathcal{I}}$. First, the measurement error vector is linearized:

$$\begin{aligned}
\boldsymbol{\omega}_{\text{err}}^{\mathcal{I}} &= \sum_n k_n \boldsymbol{v}_n^{\mathcal{I}} \times \tilde{C}^{\mathcal{I}}\boldsymbol{v}_n^{\mathcal{I}} \\
&\approx \sum_n k_n \boldsymbol{v}_n^{\mathcal{I}} \times \tilde{C}_{*i}^{\mathcal{I}}(1 + \boldsymbol{x}_\times^{\mathcal{I}})\boldsymbol{v}_n^{\mathcal{I}} \\
&= \sum_n k_n \boldsymbol{v}_n^{\mathcal{I}} \times \tilde{C}_{*i}^{\mathcal{I}}\boldsymbol{v}_n^{\mathcal{I}} - \sum_n k_n \boldsymbol{v}_n^{\mathcal{I}} \times \tilde{C}_{*i}^{\mathcal{I}}(\boldsymbol{v}_n^{\mathcal{I}})_\times \boldsymbol{x}^{\mathcal{I}} \\
&= -\left(\sum_n k_n (\tilde{C}_{*i}^{\mathcal{I}}\boldsymbol{v}_n^{\mathcal{I}})_\times (\boldsymbol{v}_n^{\mathcal{I}})_\times\right)\boldsymbol{x}^{\mathcal{I}} \\
&= \tilde{C}_{*i}^{\mathcal{I}}\sum_n k_n \left((\boldsymbol{v}_n^{\mathcal{I}})^T\tilde{C}_{*i}^{\mathcal{I}}\boldsymbol{v}_n^{\mathcal{I}} I - \boldsymbol{v}_n^{\mathcal{I}}(\boldsymbol{v}_n^{\mathcal{I}})^T\tilde{C}_{*i}^{\mathcal{I}}\right)\boldsymbol{x}^{\mathcal{I}} \\
&= \tilde{C}_{*i}^{\mathcal{I}}\left(\operatorname{tr}(M^{\mathcal{I}})I - M^{\mathcal{I}}\right)\tilde{C}_{*i}^{\mathcal{I}}\boldsymbol{x}^{\mathcal{I}} \\
&= A_i^{\mathcal{I}}\boldsymbol{x}_i^{\mathcal{I}}
\end{aligned} \tag{16}$$

where $A_i^{\mathcal{I}} = \tilde{C}_{*i}^{\mathcal{I}}\left(\operatorname{tr}(M^{\mathcal{I}})I - M^{\mathcal{I}}\right)$ and $\boldsymbol{x}_i^{\mathcal{I}} = \tilde{C}_{*i}^{\mathcal{I}}\boldsymbol{x}^{\mathcal{I}}$. Substituting (16) into (15) with the linearized $\tilde{C}^{\mathcal{I}}$ and $\tilde{\boldsymbol{b}}^{\mathcal{I}}$ results in

$$\begin{bmatrix} \dot{\boldsymbol{x}}_i^{\mathcal{I}} \\ \dot{\boldsymbol{y}}^{\mathcal{I}} \end{bmatrix} = \begin{bmatrix} -K_P^{\mathcal{I}}A_i^{\mathcal{I}} & I \\ -K_I^{\mathcal{I}}A_i^{\mathcal{I}} & \boldsymbol{\omega}_\times^{\mathcal{I}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_i^{\mathcal{I}} \\ \boldsymbol{y}^{\mathcal{I}} \end{bmatrix}. \tag{17}$$

To demonstrate the instability of the equilibria $\tilde{C}_{*i}$ for $i = 1, 2, 3$, it is necessary to show that a trajectory starting arbitrarily close to $\tilde{C}_{*i}$ (*i.e.* $|\boldsymbol{\xi}^{\mathcal{I}}|$ arbitrarily close to zero with $\boldsymbol{\xi}^{\mathcal{I}} = (\boldsymbol{x}_i^{\mathcal{I}}, \boldsymbol{y}^{\mathcal{I}})$) must eventually diverge from the compact set, which contains the equilibrium, defined by $|\boldsymbol{\xi}^{\mathcal{I}}| \leq r$ for some r chosen such that the linearization is still valid. Consider the cost function

$$s = \frac{1}{2}(\boldsymbol{x}_i^{\mathcal{I}})^T A_i^{\mathcal{I}}\boldsymbol{x}_i^{\mathcal{I}} + \frac{1}{2}(\boldsymbol{y}^{\mathcal{I}})^T(K_I^{\mathcal{I}})^{-1}\boldsymbol{y}^{\mathcal{I}} \tag{18}$$

The time derivative of s is given by

$$\begin{aligned}\dot{s} =& \frac{1}{2}\Big(-(\boldsymbol{x}_i^{\mathcal{I}})^T(A_i^{\mathcal{I}})^T(K_P^{\mathcal{I}})^T A_i^{\mathcal{I}}\boldsymbol{x}_i^{\mathcal{I}} + (\boldsymbol{y}^{\mathcal{I}})^T A_i^{\mathcal{I}}\boldsymbol{x}_i^{\mathcal{I}} \\ &- (\boldsymbol{x}_i^{\mathcal{I}})^T A_i^{\mathcal{I}} K_P^{\mathcal{I}} A_i^{\mathcal{I}}\boldsymbol{x}_i^{\mathcal{I}} + (\boldsymbol{x}_i^{\mathcal{I}})^T A_i^{\mathcal{I}}\boldsymbol{y}^{\mathcal{I}} - (\boldsymbol{y}^{\mathcal{I}})^T\boldsymbol{\omega}_\times^{\mathcal{I}}(K_I^{\mathcal{I}})^{-1}\boldsymbol{y}^{\mathcal{I}} \\ &- (\boldsymbol{x}_i^{\mathcal{I}})^T(A_i^{\mathcal{I}})^T(K_I^{\mathcal{I}})^T(K_I^{\mathcal{I}})^{-1}\boldsymbol{y}^{\mathcal{I}} + (\boldsymbol{y}^{\mathcal{I}})^T(K_I^{\mathcal{I}})^{-1}\boldsymbol{\omega}_\times^{\mathcal{I}}\boldsymbol{y}^{\mathcal{I}} \\ &- (\boldsymbol{y}^{\mathcal{I}})^T(K_I^{\mathcal{I}})^{-1}K_I^{\mathcal{I}}A_i^{\mathcal{I}}\boldsymbol{x}_i^{\mathcal{I}} + (\boldsymbol{x}_i^{\mathcal{I}})^T\dot{A}_i^{\mathcal{I}}\boldsymbol{x}_i^{\mathcal{I}} \\ &+ (\boldsymbol{y}^{\mathcal{I}})^T\frac{\mathrm{d}}{\mathrm{d}t}(K_I^{\mathcal{I}})^{-1}\boldsymbol{y}^{\mathcal{I}}\Big) \\ =& -(\boldsymbol{x}_i^{\mathcal{I}})^T(A_i^{\mathcal{I}})^T K_P^{\mathcal{I}} A_i^{\mathcal{I}}\boldsymbol{x}_i^{\mathcal{I}} \\ &- \frac{1}{2}(\boldsymbol{y}^{\mathcal{I}})^T(K_I^{\mathcal{I}})^{-1}(\dot{K}_I)^{\mathcal{I}}(K_I^{\mathcal{I}})^{-1}\boldsymbol{y}^{\mathcal{I}}\end{aligned} \tag{19}$$

which is negative definite as $\dot{K}_I$ and thus $(\dot{K}_I)^{\mathcal{I}}$ are positive definite. For $i = 1, 2, 3$, A_i has at least one negative eigenvalue, and thus there is some $\boldsymbol{\xi}_0^{\mathcal{I}}$, with magnitude arbitrarily close to zero, for which $s(\boldsymbol{\xi}_0^{\mathcal{I}}) < 0$. As $\dot{s} < 0$ and because r was chosen such that the linearization is valid, trajectories $\boldsymbol{\xi}^{\mathcal{I}}(t)$ starting at $\boldsymbol{\xi}_0^{\mathcal{I}}$ must eventually pass through the sphere with radius r. Thus, these equilibria are unstable.

The local exponential stability of the equilibrium point $(\tilde{C}, \tilde{\boldsymbol{b}}) = (I, \mathbf{0})$ is now proven. Take the $\tilde{C}_{*0} = I$ case of the linearized system (17) and return the system to body coordinates. The simplified linearized system is

$$\dot{\boldsymbol{\xi}} = B\boldsymbol{\xi}, \quad B = \begin{bmatrix} -K_P A_0 - \boldsymbol{\omega}_\times & I \\ -K_I A_0 & 0 \end{bmatrix} \tag{20}$$

where $A_0 = \mathrm{tr}(M)I - M$ is positive definite. To prove the exponential stability of the equilibrium point $\boldsymbol{\xi} = (\mathbf{0}, \mathbf{0})$ of the linearized system (20), consider the Lyapunov function candidate

$$w = \frac{1}{2}\boldsymbol{\xi}^T P\boldsymbol{\xi}, \quad P = \begin{bmatrix} A_0 & -\alpha A_0 \\ -\alpha A_0 & K_I^{-1} \end{bmatrix} \tag{21}$$

where α is chosen such that P is positive definite, or more specifically such that $\alpha^2 A_0 < K_I^{-1}$. The derivative of w along the trajectories of the system is

$$\dot{w} = \boldsymbol{\xi}^T\left(PB + B^T P + \dot{P}\right)\boldsymbol{\xi} = -\boldsymbol{\xi}^T Q\boldsymbol{\xi}$$

where

$$Q = -(PB + B^T P + \dot{P})$$
$$= \begin{bmatrix} 2A_0(K_P - \alpha K_I)A_0 & -\alpha(A_0 K_P A_0 - A_0 \boldsymbol{\omega}_\times) \\ -\alpha(A_0 K_P A_0 + \boldsymbol{\omega}_\times A_0) & 2\alpha A_0 + K_I^{-1}\dot{K}_I K_I^{-1} \end{bmatrix}$$

Using the Schur complement condition for positive definiteness, Q is positive definite exactly when $K_P > \alpha K_I$ and

$$2\alpha A_0 + K_I^{-1}\dot{K}_I K_I^{-1}$$
$$> \frac{1}{2}\alpha^2 (A_0 K_P + \boldsymbol{\omega}_\times)(K_P - \alpha K_I)^{-1}(K_P A_0 - \boldsymbol{\omega}_\times)$$

With $\dot{K}_I \geq 0$, it is straightforward to show that there exists some $\alpha > 0$ such that $P > 0$ and $Q > 0$. Therefore, $\dot{w}$ is upper bounded by a negative constant and the linearized system is exponentially stable.

Together, the results on the asymptotic convergence of $(\tilde{C}, \tilde{\boldsymbol{b}})$ to $(\tilde{C}_{*i}, \mathbf{0})$, the instability of the $\tilde{C}_{*1}$, $\tilde{C}_{*2}$, and $\tilde{C}_{*3}$ equilibria, and the stability of the $\tilde{C}_{*0}$ equilibrium show the asymptotic stability of $(I, \mathbf{0})$ with domain of attraction $\mathcal{D}$. □

A direct corollary of Theorem 1 is that the deterministic [21] bias-free MEKF is almost globally asymptotically stable. Simply identify $K_P \equiv P_a$ and set the bias error in the proof to zero. It is easy to show, given the boundedness of $\boldsymbol{\omega}$, that P_a and $\dot{P}_a$ meet the boundedness requirements of K_P and $\dot{K}_P$ for the theorem. The proof then proceeds without modification. Another interesting special case is that of the deterministic constant gain MEKF. The following theorem demonstrates that the deterministic constant gain MEKF is almost globally asymptotically stable as well.

Theorem 2. *The deterministic constant gain MEKF defined by (5) with* $P_a = P_a(\infty)$ *and* $P_c = P_c(\infty)$ *is asymptotically stable with a domain of attraction* $\mathcal{D}$ *defined in Theorem 1.*

Proof. The proof proceeds by demonstrating that the constant gain MEKF is a special case of the generalized attitude filter. It has already been shown that the forms for the MEKF and the generalized attitude filters are similar. All that remains to be shown is the positive definite nature of the matrix gains, P_a and $-P_c$ in the constant gain MEKF.

With $\boldsymbol{\omega} = \mathbf{0}$ and assuming the covariance matrix converges then equations for the matrix gains are

$$\dot{P}_a = -2\mathbb{P}_s(P_c) + \sigma_\omega^2 I - P_a A_0 P_a = 0 \tag{22a}$$

$$\dot{P}_b = \sigma_b^2 I - P_c^T A_0 P_c = 0 \tag{22b}$$

$$\dot{P}_c^T = -P_b - P_c^T A_0 P_a = 0 \tag{22c}$$

where as in Theorem 1, $A_0 = \text{tr}(M)I - M$. The combining of (22b) and (22c) yields

$$P_c = -\sigma_b^2 P_a P_b^{-1}. \tag{23}$$

Substitution into (22a) results in

$$P_a P_b^{-1} + P_b^{-1} P_a = -\sigma_b^{-2}\sigma_\omega^2 I + P_b^2$$

from which it is evident that P_a and P_b are simultaneously diagonalizable. Thus, as P_a and P_b are positive definite since the entire covariance matrix is positive definite, (23) shows that P_c is negative definite.

Identifying $K_P \equiv P_a$ and $K_I \equiv -P_c$ results in the equations for the generalized attitude filter, which by Theorem 1 is asymptotically stable with a domain of attraction $\mathcal{D}$. $\square$

V. Conclusions

This paper introduces a very general class of attitude estimators that contains Mahony's explicit nonlinear complementary filter, the bias-free multiplicative extended Kalman filter (MEKF), and the constant-gain MEKF as special cases. This generalized attitude filter is a modification of Mahony's filter that simply replaces the constant scalar gains by potentially time-varying matrix gains. The stability proofs developed for Mahony's filters extend naturally to the generalized filter. Thus, it is possible to prove the almost global asymptotic stability of this general class of filters, and consequently provide proof of the almost global asymptotic stability of a few special cases of the MEKF. This generalized attitude filter gives the filter designer an enormous space to tune and optimize the filter, while ensuring stability. It is important to note that this generalized attitude filter does not include as a special case the full MEKF, which may not necessarily have a positive definite integral gain.

Table 1 Notation

Notation	Interpretation
a	True value
$\hat{a}$	Estimated value
$\breve{a}$	Measured value
$\tilde{a}$	Error between estimated and true value
$\langle a \rangle$	Time-averaged expectation value
A	Capital letters indicate matrices
$\tilde{A}$	Left multiply error: $\tilde{A}A^T = \hat{A}^T$
A_*	Equilibrium point
$A^{\mathcal{I}}$	Matrix in inertial frame: $A^{\mathcal{I}} = CAC^T$
$\mathbb{P}_s(A)$, $\mathbb{P}_a(A)$	Symmetric or anti-symmetric projection of A
$[A, B]$, $\{A, B\}$	Commutator or anti-commutator of A and B
$\boldsymbol{x}$	Bold lowercase letters indicate vectors
$\boldsymbol{x}^{\mathcal{I}}$	Vector in inertial frame: $\boldsymbol{x}^{\mathcal{I}} = C\boldsymbol{x}$
$\boldsymbol{x}_\times$	Matrix equivalent form of the cross product

Appendix

There is an unfortunate abundance of notation employed in this paper, which results both from the nature of the work as well as an attempt to display the results from the MEKF papers and the nonlinear complementary paper in a compatible fashion. The notation most closely follows that in Mahony's paper [13], with only minor changes made to avoid conflicts with the other works. Table 1 describes the major elements of the notation.

This work also makes heavy use of a few identities and definitions described below. The matrix equivalent form of the cross product, also known as the cross operator, is defined by

$$\boldsymbol{x}_\times \equiv \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

The symmetric and anti-symmetric matrix operators are defined as

$$\mathbb{P}_s(A) \equiv \frac{1}{2}(A + A^T)$$
$$\mathbb{P}_a(A) \equiv \frac{1}{2}(A - A^T).$$

The commutator and anti-commutator are defined as

$$[A, B] \equiv AB - BA$$

$$\{A, B\} \equiv AB + BA.$$

Some useful identities for manipulating cross products and cross operators are

$$(\boldsymbol{x} \times \boldsymbol{y})_\times = [\boldsymbol{x}_\times, \boldsymbol{y}_\times] \tag{24a}$$

$$Q\boldsymbol{x} \times Q\boldsymbol{y} = (\det Q)Q^{-1}(\boldsymbol{x} \times \boldsymbol{y}) \tag{24b}$$

$$\boldsymbol{x}^T\boldsymbol{y} = -\frac{1}{2}\mathrm{tr}\left(\boldsymbol{x}_\times\boldsymbol{y}_\times\right) \tag{24c}$$

$$\boldsymbol{x}_\times\boldsymbol{y}_\times = \boldsymbol{y}\boldsymbol{x}^T - \boldsymbol{y}^T\boldsymbol{x}I \tag{24d}$$

where Q in (24b) must be positive or negative definite. The following identities are useful for expanding or contracting matrices with the cross operator:

$$\mathbb{P}_a(A\boldsymbol{x}_\times B) = \frac{1}{2}(C\boldsymbol{x})_\times \tag{25a}$$

$$S\boldsymbol{x}_\times S^T = (S\boldsymbol{x})_\times \tag{25b}$$

$$A\boldsymbol{x}_\times A = (D\boldsymbol{x})_\times \tag{25c}$$

where $A = S\Lambda_A S^T$, $B = S\Lambda_B S^T$, $C = S\Lambda_C S^T$, and $D = S\Lambda_D S^T$ with $S \in SO(3)$ and where

$$\Lambda_A = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$$

$$\Lambda_B = \mathrm{diag}(\lambda_1', \lambda_2', \lambda_3')$$

$$\Lambda_C = \mathrm{diag}(\lambda_2\lambda_3' + \lambda_2'\lambda_3, \lambda_1\lambda_3' + \lambda_1'\lambda_3, \lambda_1\lambda_2' + \lambda_1'\lambda_2)$$

$$\Lambda_D = \mathrm{diag}(\lambda_2\lambda_3, \lambda_1\lambda_3, \lambda_1\lambda_2).$$

Identities (25b) and (25c) are just special cases of the first identity (25a).

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